Re-Exam - Statistics (WBMA009-05) 2025/2026

Date and time: January 30, 2026, 15.00-17.00h
Place: Exam Hall 1, Blauwborgje 4

Rules to follow:

- This is a closed book exam. Consultation of books and notes is **not** permitted. You may use a simple (non-programmable) calculator.
- Write your name and student number on each sheet of paper.
- There are four exercises worth a total of 36 points.
- Your exam grade will be computed as follows:

$$\text{grade} = (\text{number of points} + 4) \cdot 0.25$$

  rounded to the nearest 0.5, with the usual exception that 5.5 is not possible.
- ALWAYS include the relevant equation(s) and/or short derivations.
- **We wish you success in completing the exam!**

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α	0.025	0.05	0.1	0.5	0.9	0.95	0.975
$\mathcal{N}(0,1)$	-1.96	-1.64	-1.28	0	1.28	1.64	1.96

Table 1: Quantiles q_α of the $\mathcal{N}(0,1)$ distribution.

1. **Geometric distribution.** **10**
Consider a random sample of size n from the geometric distribution

$$X_1, \ldots, X_n \sim \text{Geo}(\theta)$$

with parameter $0 < \theta < 1$ and density:

$$f_\theta(x) = (1-\theta)^{x-1} \cdot \theta \qquad (x = 1, 2, 3, \ldots)$$

It can be shown that this implies the expectation $E[X_1] = \frac{1}{\theta}$.

(a) Compute the ML estimator of θ. 2
HINT: Check via the 2nd derivative whether you have a maximum.

(b) Compute the Fisher information $I(\theta)$ for a sample of size $n = 1$. 2

For the following two exercise parts (c-d) consider the test problem

$$H_0 : \theta = 0.6 \ \textbf{vs.} \ H_1 : \theta = 0.4$$

(c) Show that a statistical test that rejects H_0 if $\sum_{i=1}^{n} X_i > k_0$, where $k_0 > 0$ is a constant, is the Neyman-Pearson UMP test for the test problem. 2

(d) Consider the Neyman-Pearson UMP test from (c) with $n = 1$ and $k_0 = 3$. Compute the test level α. 2

For the last exercise part (e) assume $n = 5$ and the five realisations:
$X_1 = 1$, $X_2 = 3$, $X_3 = 1$, $X_4 = 3$, and $X_5 = 2$.

(e) Make use of the asymptotic efficiency (normality) of the ML estimator to construct an asymptotic two-sided 80% confidence interval for θ. 2
HINT: See Table 1 for a list of quantiles.

2. **Rao–Blackwell theorem.** **6**
Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$ with $\theta > 0$ and density

$$f(x|\theta) = \frac{1}{\theta} \mathbf{1}_{(0,\theta)}(x), \qquad (x \in \mathbb{R}).$$

Define $M = \max\{X_1, \ldots, X_n\}$ and consider the estimator $T_0 = 2X_1$.

(a) Show that T_0 is an unbiased estimator of θ. 1

(b) Show that M is a sufficient statistic for θ. 2

(c) Compute the Rao–Blackwell estimator $W = \mathbb{E}[T_0|M]$ and simplify it. 3

3. **Lindley distribution.** **10**

Consider a random sample of size n from the Lindley distribution with density

$$f(x|\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \qquad (x > 0)$$

where $\theta > 0$, and denote $S := \sum_{i=1}^{n} X_i$.

(a) Show that S is a sufficient statistic for θ. 2

(b) Derive the maximum likelihood estimator of θ. 2
HINT: Check via the 2nd derivative whether you have a maximum.

(c) Consider the test problem

$$H_0 : \theta \leq \theta_0 \ \textbf{vs.} \ H_1 : \theta > \theta_0.$$

Show that the uniformly most powerful (UMP) test rejects H_0 if $S \leq c$. You do not need to compute the constant c. 2

(d) Compute the Fisher information $I(\theta)$ for a sample of size $n = 1$. 2

For the last exercise part (e) assume $n = 10$ and $\widehat{\theta}_{ML} = 2$.

(e) Make use of the asymptotic efficiency (normality) of the ML estimator to construct an asymptotic one-sided 95% confidence interval of the form $(-\infty, U]$ for θ. 2
HINT: See Table 1 for a list of quantiles.

4. **P-values and power.** **10**

Consider a random sample of size n from the exponential distribution with density

$$f(x|\theta) = \theta e^{-\theta x} \qquad (x > 0)$$

where $\theta > 0$, and the test problem

$$H_0 : \theta = \theta_0 \ \textbf{vs.} \ H_1 : \theta \neq \theta_0.$$

Define $S := \sum_{i=1}^{n} X_i$ and perform a likelihood ratio test (LRT).

(a) Derive the maximum likelihood (ML) estimator of θ. 2

(b) Derive the LRT statistic, and write it as a function of S. 2

(c) Show that LRT statistic is maximised at $S = n/\theta_0$ and decreases as S moves away from n/θ_0, so that the LRT rejects H_0 for

$$S \leq c_1 \quad \text{or} \quad S \geq c_2$$

for suitable constants $c_1 < c_2$. 4

(d) For $\theta \neq \theta_0$, write down in one line the power function of the LRT, in terms of S and the critical values c_1, c_2. 2

HINT: Recall the LRT statistic: $\dfrac{\sup_{\theta \in \Theta_0} L(\theta|x_1, \ldots, x_n)}{\sup_{\theta \in \Theta} L(\theta|x_1, \ldots, x_n)}$.